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The difference $5\frac{1}{2}$ in a problem of rations from the Rhind mathematical papyrus

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Abstract

In an ancient Egyptian problem of bread distribution from the Rhind mathematical papyrus (dated between 1794 and 1550 B.C.), a procedure of “false position” is used in the calculation of a series of five rations. The algorithm is only partially illustrated in the problem text, and last century’s prevailing interpretations suggested a determination of the series by trial and error. The missing part of the computational procedure is reconstructed in this article as an application of the algorithm, exemplified in the preceding section of the papyrus, to calculate an unknown quantity by means of the method of “false position.”

Sommario

In un antico problema Egizio di distribuzione di pane, nel papiro matematico di Rhind (datato tra il 1794 e il 1550 A.C.), una procedura di “falsa posizione” è utilizzata nel calcolo di una serie di cinque razioni. L’algoritmo è illustrato solo parzialmente nel testo del problema, e le interpretazioni prevalenti nel secolo scorso suggerivano una determinazione della serie per tentativi successivi. La parte mancante dell’algoritmo è ricostruita in questo articolo come applicazione dell’algoritmo, esemplificato nella sezione precedente del papiro, per calcolare una quantità incognita mediante il metodo di “falsa posizione.”

MSC: 01A16

Keywords: Egyptian mathematics; Rhind papyrus; Bread distribution; Method of false position; Egyptian algorithms; Embedded subalgorithms

1. Introduction: the structure of the Rhind papyrus and techniques of calculation with fractions

The Rhind mathematical papyrus, an over 5 meters long papyrus roll written on both surfaces, was found in the middle of the nineteenth century at Thebes in Upper Egypt, and has been in the British Museum (BM 10057-8) since 1865, apart from small fragments separately traded, which are now owned by the Brooklyn Museum (37.1784E). Together with the Moscow papyrus, it is the most significant extant mathematical source from ancient Egypt.[1] It

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[1] A complete analysis of the Rhind papyrus was first accomplished by Peet [1923], with hieroglyphic rewriting, translation, and commentary of the text. The papyrus was then published by Chace et al. [1927–1929], including hieratic rewriting, transcription, translation, and commentary. A more recent photoreproduction with commentary can be found in Robins and Shute [1987]. The main problem texts from the Rhind papyrus and other dynastic sources are analyzed and rewritten as symbolic algorithms by Imhausen [2003a].

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was written in hieratic—the cursive script for everyday, nonsacred purposes—during the Second Intermediate Period (1794–1550 B.C.), as a copy of an earlier papyrus. The structure of the papyrus, with indication of groups of problems classified by subject, is as follows:

Heading: title, date, name of the copyist;[2]
Opening section: 2 divided by odd numbers from 3 to 101, quotients of 1 to 9 divided by 10;[3]
Problems nos. 1–6: rations of bread;
Problems nos. 7–20: fractions multiplied by a factor;
Problems nos. 21–23: *sekem* "completion";
Problems nos. 24–34: *aha* "quantity";
Problems nos. 35–38: metrologic *aha* "quantity";
Problems nos. 39–40: rations of bread;
Problems nos. 41–46: granary capacity;
Problem no. 47: metrologic conversion table;
Problems nos. 48–55: *ahet* "area";
Problems nos. 56–60: *seked* "slope" of pyramids;
Problem no. 61b: two-thirds of a fraction;
Problem no. 62: value of metals;
Problems nos. 63–66: rations of bread, grain, fat;
Problem no. 67: *bakw* "product of working";
Problem no. 68: grain production of groups of workers;
Problems nos. 69–78: *pesw* "baking ratio";
Problem no. 79: successive multiplications by 7;
Problems nos. 80–81: metrologic conversion tables;
Problems nos. 82–84: animal feeding;
Parts nos. 85–87: non-mathematical records.

In Imhausen's classification of Egyptian problem types, problems dealing with basic mathematical techniques are distinguished from those having a practical background related to administration or construction.[4] Most problems of the Rhind papyrus are concerned with daily matters of economic activities, such as farming, baking, brewing, and the distribution of goods. In this paper I analyze Problem 40, an administrative exercise of bread distribution. The problem is based on mathematical techniques described in the preceding sections of the papyrus, in particular techniques of calculation with fractions. With the exception of $\frac{2}{3}$ and, in some limited cases, $\frac{3}{4}$, any Egyptian fraction denotes the inverse of an integer. In the standard notation introduced by Neugebauer in [1926], the overbar denotes inversion (indicated by r "part" in hieroglyphs) and the special sign for $\frac{2}{3}$ is transcribed as $\overline{\overline{3}}$. A multiple of a fraction is expressed by a sum of Egyptian fractions in descending order of magnitude, as largely displayed in the so-called "$2:n$ table" in the Opening section of the Rhind papyrus, which shows the division of 2 by odd numbers from 3 to 101. For instance, the division of two by five, or twice one-fifth, is written as $\overline{3}\,\overline{15}$. The additive process in operations of multiplication and division is typically accomplished in a scheme with two columns, one displaying appropriate multipliers (e.g., 1 and its successive doubling or halving) of the given factor/divisor, the other the results of such intermediate operations. Division is thus executed as a form of multiplication, by operating on the divisor to find the dividend. For example, as we will see, in Problem 40 the division of 100 by 60 is executed by writing 1 and 60 in a first line, indicating that 1 multiplied by 60 is 60. In a second line, the numbers $\overline{\overline{3}}$ and 40 signify that 40, which completes 60 to 100, is $\overline{\overline{3}}$ of 60. The result is hence the sum of 1 and $\overline{\overline{3}}$, i.e., $1\,\overline{\overline{3}}$.[5] The administrative Problems 1–6 from the Rhind papyrus, in which a given number of loaves is distributed among 10 men, display further results of operations

[2] The name of the copyist is given as *Iah-mesw* "Born of the Moon", usually rendered as Ahmes or Ahmose.
[3] The fragments of the papyrus in the Brooklyn Museum contain the division of 2 by 101, quotients of 1 to 9 divided by 10, and part of Problems 1–5.
[4] See Imhausen [2003b, 4–6].
[5] For a more comprehensive account of Egyptian mathematical operations, see Ritter [2000, 126–130].

as sums of fractions: for example, the division of 7 loaves by 10 is represented by $\bar{\bar{3}}\,\overline{30}$.[6] In Problems 7–20, fractions are multiplied by a factor ($1\,\bar{2}\,\bar{4}$, $1\,\bar{\bar{3}}\,\bar{3}$), and the technique of "completion to 1" of a sum of fractions is illustrated in Problems 21–23.

Before dealing with the details of Problem 40, I will now first explain two concepts that play an important role in my analysis.

2. Problems of *aha* "quantity" and the method of "false position"

Problems 24–27 and 31–34 of the Rhind papyrus concern the calculation of an unknown quantity (*ʿḥʿ*, conventionally pronounced *aha*), which, when it is added to a fraction or several fractions of itself, results in a given amount. The etymology of *aha* "quantity," from the Old Kingdom term *aha* "pile," may be indicative of early practical exemplifications making use of a number of objects, to be assembled in piles. As pointed out by Imhausen [2003a, 36–38], the interpretation of the *aha* problems as modern equations, based on an improper substitution of the unknown "x" for the term *aha*, is anachronistic and misleading. The algorithmic structure of a problem text in Egyptian mathematics is befittingly highlighted by a tool proposed by Ritter [1989], which has been applied to the main sources by Imhausen [2003a]. This tool consists in rewriting a procedure as a sequence of operations. Three types of values characterize a problem text: data, constants, and results of sequentially executed operations. An analysis of the *aha* problems in the Rhind papyrus and other sources makes it possible to identify various algorithms used for their solution.[7] In Problems 31–34, the sum of the coefficient 1 of the unknown quantity and several fractions of it assigned as data is divided into the required sum. For example, in Problem 32, a quantity, its third, and its fourth are required to add up to 2, and the solution is the division of 2 by $1\,\bar{3}\,\bar{4}$. In this two-steps algorithm, consisting of an addition and a division, the execution of the division is usually particularly awkward.[8] The procedure is different when the unknown quantity is added to only one fraction of itself (Problems 24–27). In this case, the so-called "method of false position" is used: a "false" value is guessed for the *aha* "quantity," and then the sum of this value and its fractional quantity divided into the required sum yields a factor, by which the "false" *aha* is finally multiplied to find the solution. The necessary operations are executed in sequential order, step by step. The text of Problem 26 is translated in the first column of Table 1, in which the details of the computations have been omitted and are indicated between curly braces. The data D_1 and D_2 can be identified in the initial statement: "A quantity, whose $\bar{4}$ ($= D_1$) is added, results in 15 ($= D_2$)." In the right-side columns of Table 1, the rhetorical system of instructions is rewritten as a symbolic algorithm, whose procedural steps are also identifiable in Problems 24–25 and 27.[9] The step-by-step instructions for the solution of the problem correspond to the sequence (1)–(P):

(1) A "false" value for the *aha* "quantity," namely 4, is calculated as the inverse of the fraction D_1.
(2) The fraction D_1 multiplied by the "false" value (1) gives 1.
(3) The two integers (1) and (2) are added to yield the false value plus its fourth.
(4) A factor of correction results from the division of the demanded sum D_2 by the sum (3).
(5) The factor (4) multiplied by (1) yields the true *aha* "quantity".
(6) The fraction D_1 of the *aha* "quantity" is calculated.
(P) As a proof of the correctness, the results (5) and (6) are added.

Operation (6), which gives the same result as (4), is omitted in the other three problems of the subgroup, for which the verification is performed by adding (5) and (4). Clearly, the result (4) is both the factor of correction and the fractional quantity to be added. The term "false" is used by modern authors as a synonym for a convenient

6 See Imhausen [2003b, 7–8].
7 Cf. Imhausen [2003a, 39–53].
8 The algorithm of Problems 31–34 is used also for Problem 30, in which $\bar{\bar{3}}\,\overline{10}$ of the unknown quantity equals the given value 10. See Imhausen [2003a, 212–221, 42–46].
9 For an analysis and discussion of the problems in this subgroup, see Imhausen [2003a, 206–209, 39–42]. For the hieratic text, see Robins and Shute [1987, Plates 10–11].

Table 1
pRhind, Problem 26 (my adaptation from the German translation by Imhausen [2003a])

Problem-text	Algorithm		
A quantity, whose $\overline{4}$ is added, results in 15.	Data	D_1, D_2	$\overline{4}$, 15
Calculate with 4!	(1)	$1 : D_1$	$[1 : \overline{4} =]4$
Then you calculate its $\overline{4}$ as 1.	(2)	$D_1 \cdot (1)$	$\overline{4} \cdot 4 = 1$
Sum: 5.	(3)	$(1) + (2)$	$4 + 1 = 5$
Divide 15 by 5! {Division} Then 3 results.	(4)	$D_2 : (3)$	$15 : 5 = 3$
Multiply 3 by 4! {Multiplication} Then 12 results.	(5)	$(4) \cdot (1)$	$3 \cdot 4 = 12$
{Multiplication of $\overline{4}$ by 12}	(6)	$D_1 \cdot (5)$	$\overline{4} \cdot 12 = 3$
The quantity is 12, $\overline{4}$ of it (is) 3, sum 15.	(P)	$(5) + (6) = D_2$	$12 + 3 = 15$

guess, although it is never used in Egyptian texts dealing with *aha* problems.[10] In comparison with the algorithm of Problems 31–34, the procedure of "false position" here entails a generally simpler division by an integer number.

3. Rations of bread and the procedure of "false position" in Problem 40 of the Rhind papyrus

As previously mentioned, Problems 1–6 of the Rhind papyrus concern the equal distribution to 10 men of a given amount of bread from 1 up to 9 loaves: a ration is expressed by a fraction or by a sum of fractions. In Problems 39 and 40, a given amount of bread is unequally distributed to individuals belonging to two groups.[11] Problem 39, whose title is "Method for calculating a difference," is formulated as follows:

100 loaves for 10 men, 50 for 6, 50 for 4. What is the difference?

Thus two groups of recipients are respectively composed of 6 and 4 men, and the difference between the equal rations in one group (namely, $50 : 4$) and those in the other (namely, $50 : 6$) is calculated.

In Problem 40, the distribution between the two groups is given by the condition (see the first column of Table 2)

100 loaves for 5 men, $\overline{7}$ of the 3 upper-level to the 2 lower-level men.

Thus a group of two men receive one-seventh of the loaves attributed to a group of three men. From the subsequent calculations, one deduces the further condition that the difference of any two successive rations is a constant, hence the five rations are in arithmetical progression.

The essential characteristics of the distributions in Problems 1–6 and in Problems 39 and 40 are summarized in Table 3. In Problem 39, the totals for the two groups are equal, and the rations are equal within each group, but unequal for individuals in distinct groups. In Problem 40, the distribution is entirely unequal: the totals for the two groups are in a ratio of one to seven, and the portions in the two groups differ by a constant value. In order to find the solution, a ration difference of $5\,\overline{2}$ is introduced (Table 2, first column), and then the rations are computed and added. Since the obtained sum of 60 is not equal to the demanded 100, each initially calculated ration (fourth column) is multiplied by the factor obtained from the division of 100 by 60 (third column)[12] to yield the correct values.

Thus we see that a procedure of "false position" has been used: "false" portions adding up to 60 are rectified by a conversion factor to give the demanded sum of 100. However, part of the algorithm is missing: how were the "false" rations and their difference determined? Various attempts at a reconstruction were made during the last century. Two of these assumed "trial-and-error" arrangements of the five-term series with 1 as the lowest term,[13] another the solution of two simultaneous equations,[14] an anachronistic hypothesis excluded already by Peet [1923, 78–79]. On the other

[10] The term "false" is used for a supposition in Babylonian problem texts dated to the first half of the second millennium B.C.; cf. Ritter [2004, 179]. In later periods, it occurs also in Chinese, Indian, Islamic, and medieval European mathematics.

[11] A hieroglyphic rewriting, transcription, and German translation of pRhind, nos. 39–40, can be found in Imhausen [2003a, 230–231]. An English translation is included in Imhausen [2003b, 9]. For the hieratic text, see Robins and Shute [1987, Plate 13].

[12] The typical process of division is discussed in Section 1.

[13] Gillings [1972, 170–173], Clagett [1999, 56–58].

[14] Couchoud [1993, 157–160].

Table 2
pRhind, Problem 40: problem-text (slight amendment of the translation by Imhausen [2003b])

100 loaves for 5 men, $\overline{7}$ of the 3	$\backslash \cdot 23$	$\backslash \cdot 60$	times 23.	Then $38\,\overline{3}$ results.
upper-level for the 2 lower-level men.	$\backslash \cdot 17\,\overline{2}$	$\backslash \overline{\overline{3}}\,40$	$\cdot 17\,\overline{2}$	$\cdot 29\,\overline{6}$
What is the difference?	$\backslash \cdot 12$	Then you	$\cdot 12$	$\cdot 20$
Calculation, as result of the	$\backslash \cdot 6\,\overline{2}$	multiply $1\,\overline{\overline{3}}$	$\cdot 6\,\overline{2}$	$\cdot 10\,\overline{\overline{3}}\,\overline{6}$
difference $5\,\overline{2}$:	$\backslash \cdot 1$ sum 60		1	$\cdot 1\,\overline{\overline{3}}$ sum 100

Table 3
Characteristics of bread distribution in pRhind Problems 1–6 and 39/40

	pRhind, Problems 1–6	pRhind, Problem 39	pRhind, Problem 40
Totals for the two groups	—	Equal	Unequal
Individual distribution in different groups	—	Unequal	Unequal
Individual distribution in a group	Equal	Equal	Unequal

hand, Imhausen, rewriting the problem as a symbolic algorithm, assumed that the initial series of rations was calculated on the basis of a smallest ration of 1 loaf, leaving the calculation of the "false" ration difference undetermined.[15] However, none of these earlier explanations paid sufficient attention to the condition of group distribution, which can be expressed as follows: the quantity of bread in the upper group plus $\overline{7}$ of it, pertaining to the lower group, results in 100 loaves. This is clearly an *aha* problem of the type that was solved by the method of "false position" in the preceding section of the papyrus (Problems 24–27). In fact, the fraction $\overline{7}$ is already used as a datum in Problem 24. In the following paragraph I will present an explanation for the solution of problem 40 that, taking into account mathematical techniques illustrated in the papyrus and the algorithmic structure of the problem texts, is historically more plausible than the previous ones.

4. Reconstructing the missing part of the algorithm in Problem 40

It is convenient to first note that an arithmetical progression $(a_1, a_2, a_3, a_4, a_5)$ can be written as $(a_3 - 2d, a_3 - d, a_3, a_3 + d, a_3 + 2d)$, with $a_3 = (a_1 + a_2 + a_3 + a_4 + a_5)/5$. Given the value $a_1 + a_2 + a_3 + a_4 + a_5$, and once the partial sums $a_1 + a_2$ and $a_3 + a_4 + a_5$ are known, the difference d derives by simple arithmetic from one of the following straightforward relations: $2a_3 - 3d = a_1 + a_2$; $3a_3 + 3d = a_3 + a_4 + a_5$. I will now argue that two procedures of "false position" are implied in the formulation of Problem 40, and that the missing part of the algorithm can be reconstructed as an application of the *aha* algorithm as described in Problems 24–27 (cf. Table 4):

(0) The original value of 100 is converted into a "false" aggregate of 60 loaves ($= D'_1$).[16]
(1)–(P) Given a fraction of $\overline{7}$ ($= D_3$), the *aha* procedure of "false position" attributes $7\,\overline{2}$ loaves to the lower group of 2 men, and $52\,\overline{2}$ loaves to the upper group of 3 men.
(6) A fictitious equal distribution of 12 ($= 60 : 5$) loaves per man is calculated.
(7) In the upper group, 12 loaves per man yield a total of 36 loaves.
(8) The difference between the sum of the rations in the upper group as found by means of the *aha* algorithm (namely, $52\,\overline{2}$) and (7) is $16\,\overline{2}$.
(9) The division of $16\,\overline{2}$ by the constant 3 ($= 1 + 2$, i.e. the number of differences in the sequence of rations in the upper group $(a_3, a_3 + d, a_3 + 2d)$), gives the "false" ration difference of $5\,\overline{2}$.[17]

[15] Imhausen [2003a, 102]. Some misprints in the section concerned are corrected in a review by Brack-Bernsen [2004, 229].
[16] Probably the problem was set up on the basis of a difference already calculated, by means of the algorithm here described, with reference to an arithmetical progression of five values adding up to 60. This would explain both the conversion from 100 to 60 and the partial execution of the algorithm in the problem text.
[17] Assuming 3 lower-level men and 4 upper-level men as data, the constant would be 6 ($= 1 + 2 + 3$), in accordance with the law of "triangular numbers." The calculation of triangular numbers (sum of the numbers from 1 to 10, or 10th triangular number, and presumably the sum of the first 10 triangular numbers) is the subject of a problem in the demotic papyrus BM 10520 from the Roman period; see Couchoud [1986].

Table 4
Reconstruction of the missing procedure in pRhind, Problem 40

Data	D_1	100
	D_2	5
	D_3	$\overline{7}$
	D_4	3
	D_5	2
(0)	$D_1 \rightarrow D_1'$	$100 \rightarrow 60$
(1)	$1 : D_3$	$1 : \overline{7} = 7$
(2)	$D_3 \cdot (1)$	$\overline{7} \cdot 7 = 1$
(3)	$(1) + (2)$	$7 + 1 = 8$
(4)	$D_1' : (3)$	$60 : 8 = 7\,\overline{2}$
(5)	$(4) \cdot (1)$	$7\,\overline{2} \cdot 7 = 52\,\overline{2}$
(P)	$(5) + (4) = D_1'$	$52\,\overline{2} + 7\,\overline{2} = 60$
(6)	$D_1' : D_2$	$60 : 5 = 12$
(7)	$(6) \cdot D_4$	$12 \cdot 3 = 36$
(8)	$(5) - (7)$	$52\,\overline{2} - 36 = 16\,\overline{2}$
(9)	$(8) : 3 = \text{Difference}^*$	$16\,\overline{2} : 3 = 5\,\overline{2}$

The underlying idea is as follows: five rations differing by a constant value can be derived from an equal distribution to the five recipients by adding a value *three times* to the upper group and subtracting it *three times* from the lower group. With the difference 1, for instance, we obtain 10 ($= 12 - 1 - 1$), 11 ($= 12 - 1$), 12, 13 ($= 12 + 1$), 14 ($= 12 + 1 + 1$).[18] The ration difference for the demanded group distribution is hence one-third of the difference between the sum of the rations in the upper group as yielded by the *aha* algorithm ($52\,\overline{2}$) and the sum of the rations in the upper group in equal distribution (36), i.e., $5\,\overline{2}$. Alternatively, the ration difference can be determined "specularly" with reference to the lower group, as the difference between the sum of the rations in equal distribution ($24 = 12 + 12$) and the sum of the rations as yielded by the *aha* algorithm ($7\,\overline{2}$). The result is again $16\,\overline{2}$, to be divided by the constant 3 as above.[19]

5. The algorithmic structure of Problem 40 and the concept of "embedded" subalgorithms

Table 5 shows the reconstructed part of the algorithm in Problem 40 as an application of the *aha* procedure of "false position": the algorithm in Problems 24–27 is "embedded" in that for Problem 40 as subalgorithm (1)–(P). In the remaining sequence (6)–(9) of the algorithm, step (6) reproduces the determination of equal distribution as found in Problems 1–6. According to my interpretation, the complete algorithm of Problem 40, comprising the symbolic rewriting of the text (see Table 6) and the reconstructed part, consists of four stages:

- Initialization of a "false" total amount of 60 loaves instead of 100: operation (1) in Table 6.
- Calculation of the "false" ration difference: sequence (1)–(9) in Tables 4 and 5.
- Calculation of five rations adding up to 60 and differing by the value determined in the previous stage: sequence (3a)–(P′) in Table 6.
- Rectification of the rations to their correct values by means of the factor $1\,\overline{\overline{3}}$, with a verification that the sum is 100: sequence (4)–(P″) in Table 6.

[18] This process is similar to the construction of a staircase, in which the two lower-level steps are obtained by removing from a large stone three equal blocks (or two blocks of which one is twice the size of the other), which are then used to assemble the two upper-level steps. It is possible that this method, comparable to the extraction of the blocks used for aboveground courses in pyramid construction from underground rock layers, was employed in some cases to construct small rock-cut staircases.

[19] The proposed algorithm yields the following constant differences in series of five numbers adding up to 60 when fractions other than $\overline{7}$ are used: $\overline{2} \rightarrow 1\,\overline{3}$; $\overline{3} \rightarrow 3$; $\overline{4} \rightarrow 4$; $\overline{5} \rightarrow 4\,\overline{\overline{3}}$; $\overline{6} \rightarrow 5\,\overline{7}$; $\overline{8} \rightarrow 5\,\overline{2}\,\overline{3}$; $\overline{9} \rightarrow 6$. It is possible that a table containing several of these differences was available, considering that the number 60 has many divisors and its use is in general convenient when alternative values are assigned as a given fraction.

Table 5
Embedded subalgorithm

Reconstruction of missing procedure in pRhind, Problem 40		pRhind, Problems 24–25/27	
	D_1		
	D_2		
	D_3		D_1
	D_4		
	D_5		
(0)	$D_1 \to D_1'$		D_2
(1)	$1 : D_3$	(1)	$1 : D_1$
(2)	$D_3 \cdot (1)$	(2)	$D_1 \cdot (1)$
(3)	$(1) + (2)$	(3)	$(1) + (2)$
(4)	$D_1' : (3)$	(4)	$D_2 : (3)$
(5)	$(4) \cdot (1)$	(5)	$(4) \cdot (1)$
(P)	$(5) + (4) = D_1'$	(P)	$(5) + (4) = D_2$
(6)	$D_1' : D_2$		
(7)	$(6) \cdot D_4$		
(8)	$(5) - (7)$		
(9)	$(8) : 3$		

Table 6
pRhind, Problem 40, algorithm

Data	D_1	100
	D_2	5
	D_3	$\bar{7}$
	D_4	3
	D_5	2
(1)	$D_1 \to D_1'$	$[100 \to 60]$
(2)	Difference*	$5\,\bar{2}$
(3a)	$D_1' : D_2$	$[60 : 5 =]12$
(3b)	$(3a) + (2)$	$[12 + 5\,\bar{2} =]17\,\bar{2}$
(3c)	$(3b) + (2)$	$[17\,\bar{2} + 5\,\bar{2} =]23$
(3d)	$(3a) - (2)$	$[12 - 5\,\bar{2} =]6\,\bar{2}$
(3e)	$(3d) - (2)$	$[6\,\bar{2} - 5\,\bar{2} =]1$
(P′)	$(3c) + (3b) + (3a) + (3d) + (3e) = D_1'$	$23 + 17\,\bar{2} + 12 + 6\,\bar{2} + 1 = 60$
(4)	$D_1 : D_1'$	$100 : 60 = 1\,\bar{\bar{3}}$
(5a)	$(4) \cdot (3c)$	$1\,\bar{\bar{3}} \cdot 23 = 38\,\bar{3}$
(5b)	$(4) \cdot (3b)$	$1\,\bar{\bar{3}} \cdot 17\,\bar{2} = 29\,\bar{6}$
(5c)	$(4) \cdot (3a)$	$1\,\bar{\bar{3}} \cdot 12 = 20$
(5d)	$(4) \cdot (3d)$	$1\,\bar{\bar{3}} \cdot 6\,\bar{2} = 10\,\bar{\bar{3}}\,\bar{6}$
(5e)	$(4) \cdot (3e)$	$1\,\bar{\bar{3}} \cdot 1 = 1\,\bar{\bar{3}}$
(P″)	$(5a) + (5b) + (5c) + (5d) + (5e) = D_1$	$38\,\bar{3} + 29\,\bar{6} + 20 + 10\,\bar{\bar{3}}\,\bar{6} + 1\,\bar{\bar{3}} = 100$

With the execution of the *aha* algorithm imported from Problems 24–27, a second procedure of "false position" is "nested" within the first one.[20] In fact, several cases of "embedded" subalgorithms, with increasing complexity, can be found in the Rhind papyrus. For example, in the metrologic *aha* problems 35 and 38, the *aha* algorithm of

[20] The "nested" procedure provisionally assigns the value 8 to the total amount of loaves. The successive values of the total that are at some point assigned in the algorithm are thus 100; 60; 8 ($= 7 + 1$); 60 ($= 52\,\bar{2} + 7\,\bar{2}$); 100.

Problems 31–34 is "embedded" as a subalgorithm and extended by transformations of the provisional *aha* results into metrologic units.[21]

6. Conclusions

The reconstruction that I have proposed of the missing part of the algorithm for Problem 40 of the Rhind papyrus, as an application of the *aha* algorithm exemplified in Problems 24–27, is certainly more plausible than previous interpretations based on an accidental determination of the series. According to my explanation, the algorithm of Problem 40 includes implicitly the solution of an *aha* problem: after the "false" supposition of a total of 60 loaves, "embedding" and extension of the *aha* procedure of "false position" yields the "false" ration difference $5\,\overline{2}$, from which five rations adding up to 60 are computed and finally rectified to give the original sum. Considering the undoubtedly pedagogical character of the papyrus, it is possible that the task of executing the missing procedure was assigned to trainees. However, the (implicit) external execution of part of the algorithm in Problem 40 is to be traced back to the more general phenomenon, also found in Mesopotamian mathematics, of "embedded" subalgorithms, constituting a typical response to increasing complexity in mathematical languages characterized by instructions in sequential order.

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[21] On parallel structures and subalgorithms, see Imhausen [2003a, 186–189]. "Embedded" subalgorithms are also identified by Ritter [2004, 183] in Babylonian problem texts of the first half of the second millennium B.C.